1. L-1

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B.Sc - III

MATHEMATICS HONS : Paper- V

group- B. (Multiple Integrals)

Contents :→ Double and triple integrals, iterated integrals, change of order of integration.

Iterated integral :→ The integral

$$\int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$

is known as an iterated integral, because it is obtained by carrying out the one-variable integration process twice.

Remark :→

$$\int_{x=a}^{x=b} \underbrace{\boxed{\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\, dy}}_{\text{1st evaluate}} dx$$

Q1 If $f(x,y) = k$ and $R = [a,b] \times [c,d]$, find

$$\iint_R f(x,y)\, dA$$

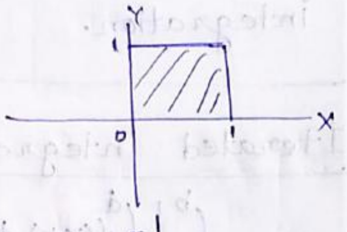
Solution :→

$$\because \iint_R f(x,y)\, dA = \iint_R f(x,y)\, dx\, dy = \int_{y=c}^{y=d} \int_{x=a}^{x=b} k\, dx\, dy$$

$$= \int_{y=c}^{y=d}\left[\int_{x=a}^{b} K\,dx\right]dy = \int_{y=c}^{d}\left\{K[x]_a^b\right\}dy$$

$$= K(b-a)\int_{y=c}^{d} dy = K(b-a)(d-c) \quad \underline{\text{Ans.}}$$

Example (2):- Find the integral $\iint_R (1-x)\,dx\,dy$ where the region R is $[0,1]\times[0,1]$



Solution: →

$$\iint_R (1-x)\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{1}(1-x)\,dx\,dy = \int_{y=0}^{y=1}\left[x-\frac{x^2}{2}\right]_0^1 dy$$

$$= \int_{y=0}^{y=1}\frac{1}{2}\,dy = \frac{1}{2}[y]_0^1 = \frac{1}{2} \quad \underline{\text{Ans.}}$$

Example (3) :→ Let $z=f(x,y)=x^2+y^2$ and Let $R=[-1,1]\times[0,1]$. Evaluate the integral $\iint_R (x^2+y^2)\,dx\,dy$

Solution: →

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_{y=0}^{y=1}\left[\int_{-1}^{1}(x^2+y^2)\,dx\right]dy \quad \text{—(1)}$$

To find $\int_{-1}^{1}(x^2+y^2)\,dx$, we treat y as constant and integrate w.r.to x.

$$\int_{-1}^{1}(x^2+y^2)\,dx = \left[\frac{x^3}{3} + y^2x\right]_{-1}^{1} = \frac{2}{3} + 2y^2$$

putting this value in eq^n (1)

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{y=0}^{y=1}\left(\frac{2}{3} + 2y^2\right)dy$$

$$= \left[\frac{2}{3}y + \frac{2}{3}y^3\right]_0^1$$

$$= \frac{4}{3} \text{ Ans.}$$

Example (4) :→ Compute $\iint_S \cos x \sin y \, dx\,dy$, where S is the square $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

Solution :→

$$\iint_S \cos x \sin y \, dx\,dy = \int_0^{\pi/2}\left[\int_0^{\pi/2} \cos x \sin y \, dx\right] dy$$

$$= \int_0^{\pi/2} \sin y \left[\int_0^{\pi/2} \cos x \, dx\right] dy$$

$$= \int_0^{\pi/2} \sin y \, dy = 1$$

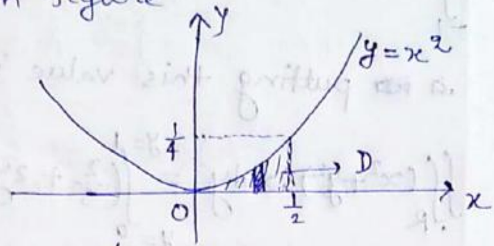
Ans.

Example (5) Compute $\iint_R (x^2+y)\,dx\,dy$, where R is the square $[0,1]\times[0,1]$

Solution :- $\int_0^1\int_0^1 (x^2+y)\,dy\,dx = \int_0^1 \left[x^2y + \frac{y^2}{2}\right]_{y=0}^{1} dx$

$$= \int_0^1 \left(x^2 + \frac{1}{2}\right)dx = \left[\frac{x^3}{3} + \frac{x}{2}\right]_0^1 = \frac{5}{6} \text{ Ans.}$$

Example(6) :- Find $\iint_D (x+y)\,dx\,dy$, where D is the shaded region in figure



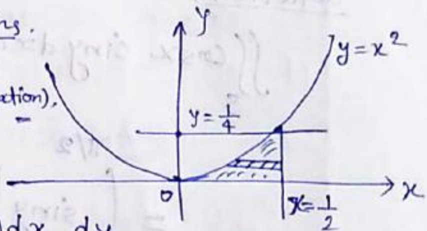
Solution :→ $\iint_D (x+y)\,dx\,dy = \int_{x=0}^{\frac{1}{2}} \int_{y=0}^{x^2} (x+y)\,dy\,dx$

$$= \int_{x=0}^{\frac{1}{2}} \left[xy + \frac{y^2}{2} \right]_{y=0}^{x^2} dx$$

$$= \int_{x=0}^{\frac{1}{2}} \left(x^3 + \frac{x^4}{2} \right) dx = \left[\frac{x^4}{4} + \frac{x^5}{10} \right]_0^{1/2}$$

$$= \frac{1}{64} + \frac{1}{320} = \frac{3}{160} \text{ Ans.}$$

2nd Method :- (change of order of integration).



$$\therefore \iint_D (x+y)\,dx\,dy = \int_{y=0}^{y=\frac{1}{4}} \int_{x=\sqrt{y}}^{x=\frac{1}{2}} (x+y)\,dx\,dy$$

$$= \int_{y=0}^{y=\frac{1}{4}} \left[\frac{x^2}{2} + yx \right]_{x=\sqrt{y}}^{\frac{1}{2}} dy = \int_{y=0}^{y=\frac{1}{4}} \left(\frac{1}{8} + \cancel{\frac{1}{2}y} - \cancel{\frac{y}{2}} - y^{\frac{3}{2}} \right) dy$$

$$= \left[\frac{1}{8}y - \frac{2}{5} y^{5/2} \right]_{y=0}^{\frac{1}{4}} = \frac{1}{8} \times \frac{1}{4} - \frac{2}{5} \times \left(\frac{1}{4}\right)^{5/2}$$

$$= \frac{1}{32} - \frac{\cancel{2}}{5} \times \frac{1}{\cancel{32}\,16} = \frac{1}{32} - \frac{1}{80} = \frac{5-2}{160} = \frac{3}{160} \text{ Ans.}$$